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Observation and Prediction for the Heat Equation, II*†

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1. INTRODUCTION

Consider an insulated uniform body occupying a bounded open subset $\mathscr{R}$ of R^m with piecewise smooth boundary $\partial\mathscr{R}$. Under proper normalization the temperature $u = u(t, x)$ satisfies

$$u_t = \Delta u, \qquad t > 0, \qquad x \in \mathscr{R}, \tag{1a}$$

$$u_n \,|_{\partial\mathscr{R}} = 0, \qquad t > 0, \tag{1b}$$

where u_n denotes the (outward) normal derivative defined at all points of $\partial\mathscr{R}$ except the negligible subset for which a normal is not defined. Suppose it possible to *observe* the boundary value $f(t, \cdot) = u(t, \cdot)|_{\partial\mathscr{R}}$ throughout the time interval $0 < t < T$. Is it then possible, given this data alone, to *predict* $w(x) = u(T, x)$, $x \in \mathscr{R}$, and furthermore is the problem "well posed"? More precisely, we ask whether the mapping

$$A: f \mapsto w \tag{2}$$

from the linear manifold $\mathscr{M}$ of boundary values assumed by solutions of (1) to the space $L^2(\mathscr{R})$, is well-defined and continuous, topologizing $\mathscr{M}$ by the norm of $L^2((0, T) \times \partial\mathscr{R})$.

In our previous paper [6] we succeeded in showing that the answer is affirmative whenever $\mathscr{R}$ is a *cylindrical* region, i.e., when $\mathscr{R} = (a, b) \times \mathscr{D}$ for some bounded open subset $\mathscr{D}$ of R^{m-1} with piecewise smooth boundary. The purpose of the present note is, first, to show that the answer is likewise

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affirmative when $\mathscr{R}$ is the m-ball and, second, to discuss the problems involved in extending these results to a more general class of regions $\mathscr{R}$.

As in the paper [6], the argument proceeds by using separation of variables to reduce this problem to a problem in the theory of approximation by exponential sums (Dirichlet series). The significant new aspect of the present problem is the necessity of developing *uniform* results concerning Dirichlet series approximation: Results which are valid for every series whose exponent sequence $\Lambda = \{\lambda_1, \lambda_2, ...\}$ belongs to a prescribed class of sequences. As far as we know, such uniformity problems involving a *class* of exponent sequences have not previously been considered (except *en passant* in Ref. [1]). For this reason, our results in this direction are presented in somewhat greater generality than is needed directly for our application (see Section 4).

2. We begin by noting that the mapping given in (2) is well-defined for every bounded $\mathscr{R}$ with piecewise smooth boundary and every $T > 0$. This follows from the fact, implied by the unique extension property for the heat equation, that the only solution to (1) which satisfies the additional condition

$$u(t, x) = 0, \qquad 0 < t < T, \qquad x \in \partial\mathscr{R}$$

is $u \equiv 0$; see, e.g., Ref. [5].

We now restrict ourselves to an investigation of the mapping A for the case of an m-ball, which without loss of generality we may take to be of radius 1. The separability of the Laplacian for the m-ball ensures that we may write any solution of (1) in the form

$$u(t, x) = \sum_{i,n} c_{in} e^{-\lambda_{in} t} \Omega_i(\omega) R_{in}(r), \tag{3}$$

with $t > 0$, ω varying over the unit sphere, and $r \in [0, 1]$, so $x = r\omega \in \bar{\mathscr{R}}$. (We refer to r as the 'radial' variable, ω as the 'angular' variable.) The $\{\Omega_i\}$ are the orthonormalized 'angular' eigenfunctions of Δ for $\mathscr{R}$, the $\{R_{in}\}$ are the corresponding radial eigenfunctions for the Neumann boundary condition (1b), and the $\{\lambda_{in}\}$ are the eigenvalues of Δ pertaining to this boundary condition. In writing (3) we have used the fact that in R^m,

$$\Delta = r^{1-m} \frac{\partial}{\partial r} \left[r^{m-1} \frac{\partial}{\partial r} \right] + r^{-2} L,$$

where L is a partial differential operator involving only the 'angular' variable ω. It follows that

$$L\Omega_i = \nu_i^2 \Omega_i, \qquad i \geqslant 1. \tag{4}$$

(Note that the operator L has multiple eigenvalues so each *distinct* eigenvalue may appear several times in the sequence $\{\nu_i^2\}$.) Likewise

$$R''_{in} + \frac{m-1}{r} R'_{in} + \left(\lambda_{in} - \frac{\nu_i^2}{r^2}\right) R_{in} = 0, \tag{5}$$
$$R'_{in}(1) = 0, \qquad R_{in}(0+) \text{ bounded.}$$

We also have the normalization conditions

$$\langle \Omega_i , \Omega_j \rangle_{\partial\mathscr{R}} = \int_{\partial\mathscr{R}} \Omega_i \Omega_j \, d\omega = \delta_j^i, \tag{6a}$$

$$\langle R_{in} , R_{in'} \rangle = \int_0^1 R_{in}(r)\, R_{in'}(r)\, r^{m-1}\, dr = 0, \qquad n \neq n', \tag{6b}$$

$$R_{in}(1) = 1, \tag{6c}$$

and we set

$$N_{in}^2 = \int_0^1 | R_{in}(r)|^2 r^{m-1} dr. \tag{6d}$$

With the above normalizations, the coefficients $\{c_{in}\}$ satisfy

$$\sum_{i,n} | c_{in} |^2 N_{in}^2 e^{-2\lambda_{in} t} < \infty, \qquad \text{for } t > 0, \tag{7}$$

since $u(t, \cdot) \in L^2(\mathscr{R})$ for all $t > 0$.

From the expansion (3) and from (6c) and (7) we see that

$$f(t, x) = \sum_i \left(\sum_n c_{in} e^{-\lambda_{in} t}\right) \Omega_i(\omega), \qquad \begin{array}{l} x = \omega \in \partial\mathscr{R}, \\ 0 < t < T, \end{array} \tag{8a}$$

$$w(x) = \sum_{i,n} (c_{in} e^{-\lambda_{in} T})\, \Omega_i(\omega)\, R_{in}(r), \qquad x = r\omega \in \mathscr{R}. \tag{8b}$$

Using (6a), we have $f(t, x) = \sum_i f_i(t)\, \Omega_i(\omega)$, with

$$f_i(t) = \sum_n c_{in} e^{-\lambda_{in} t} = \langle f(t, \cdot), \Omega_i \rangle_{\partial\mathscr{R}} = \int f(t, \cdot)\, \Omega_i \, d\omega, \tag{9}$$
$$0 < t \leqslant T, \qquad i = 1, 2, \ldots .$$

We introduce the following notation: For any sequence $\Lambda = (\lambda_1 , \lambda_2 , \ldots)$ with $0 < \lambda_1 < \lambda_2 < \ldots$, we write

$$[\Lambda] = \overline{sp}\{e^{-\lambda t} : \lambda \in \Lambda\} \subset L^2(0, \infty),$$
$$[\Lambda]_T = \overline{sp}\{e^{-\lambda t} : \lambda \in \Lambda\} \subset L^2(0, T).$$

Then let $\{l_n = l_n^{\Lambda} : n \geqslant 1\}$ and $\{l_n^T = l_n^{\Lambda,T} : n \geqslant 1\}$ denote linear functionals —assuming they are well defined—such that

$$l_n : \varphi \mapsto b_n \qquad \text{whenever} \qquad \varphi = \sum b_k e^{-\lambda_k t} \in [\Lambda] \subset L^2(0, \infty),$$

$$l_n^T : \psi \mapsto b_n \qquad \text{whenever} \qquad \psi = \sum b_k e^{-\lambda_k t} \in [\Lambda]_T \subset L^2(0, T).$$

We will be concerned, in particular, with the eigenvalue sequences $\Lambda_i = \{\lambda_{in}\}_{n \geqslant 1}$ obtained from the boundary value problem (5). In terms of the above notation and using Eq. (9), we may express the coefficients $\{c_{in}\}$ appearing in (3) and (8) by

$$c_{in} = l_{in}^T(f_i) = l_{in}^T(\langle f, \Omega_i \rangle_{\partial \mathscr{R}}), \qquad i, n \geqslant 1,$$

where for brevity we have written l_{in}^T for $l_n^{\Lambda_i, T}$. Thus (8b) can be written as

$$[Af](x) = w(x) = \sum_{i,n} l_{in}^T(\langle f, \Omega_i \rangle_{\partial \mathscr{R}})\, \Omega_i(w)\, R_{in}(r)\, e^{-\lambda_{in} T}, \quad x = r\omega \in \mathscr{R}. \tag{10}$$

It follows that if each of the $\{l_{in}^T\}$ is continuously extendable to all of $[\Lambda_i]_T \subset L^2(0, T)$ and if we can obtain suitable estimates for the $\| l_{in}^T \|$, then we can estimate $\| A \|$. In fact, we have from (6) and (10) that

$$\begin{aligned} | w |_{\mathscr{R}}^2 &= \sum_{i,n} | l_{in}^T(\langle f, \Omega_i \rangle)_{\partial \mathscr{R}} |^2 N_{in}^2 e^{-2\lambda_{in} T} \\ &\leqslant \sum_{i,n} \| l_{in}^T \|^2 |\langle f, \Omega_i \rangle_{\partial \mathscr{R}} |_{(0,T)}^2 N_{in}^2 e^{-2\lambda_{in} T} \\ &\leqslant \sup_i \left\{ \sum_n \| l_{in}^T \|^2 N_{in}^2 e^{-2\lambda_{in} T} \right\} | f |_{(0,T) \times \partial \mathscr{R}}^2, \end{aligned}$$

since by (6a) and the completeness of the $\{\Omega_i\}$,

$$\sum_i |\langle f, \Omega_i \rangle|_{(0,T)}^2 = | f |_{(0,T) \times \partial \mathscr{R}}^2 .$$

(For convenience we will consistently use $|\ |$, with subscripts for the range of integration, to denote integral L^2 norms of functions, and we will use $\|\ \|$, without subscripts, to denote operator norms of linear functionals and transformations.) Hence, we have the estimate

$$\| A \|^2 \leqslant \sup_i \left\{ \sum_n \| l_{in}^T \|^2 N_{in}^2 e^{-2\lambda_{in} T} \right\}. \tag{11}$$

By the Hahn–Banach theorem, the functional $l_n^T = l_n^{\Lambda,T}$ (defined initially on $sp\{e^{-\lambda t} : \lambda \in \Lambda\}$) is continuously extendable to $[\Lambda]_T$ if and only if the function $e^{-\lambda_n t}$, $0 < t < T$ satisfies

$$e^{-\lambda_n t} \notin [\Lambda - \{\lambda_n\}]_T = \overline{sp}\{e^{-\lambda_k t} : k \neq n\} \subset L^2(0, T). \tag{12a}$$

In this case,

$$\begin{aligned} \| l_n^T \|^{-1} &= \inf[| e^{-\lambda_n t} - \varphi |_{(0,T)} : \varphi \in sp\{e^{-\lambda_k t} : \lambda_k \in \Lambda, k \neq n\}] \\ &= \operatorname{dist}(e^{-\lambda_n t}, [\Lambda - \{\lambda_n\}]_T). \end{aligned} \tag{12b}$$

Thus, our prediction problem reduces to obtaining suitable estimates, *uniform in i*, of the quantities $\{N_{in}\}$ and

$$\| l_{in}^T \|^{-1} = \operatorname{dist}(e^{-\lambda_{in} t}, [\Lambda_i - \lambda_{in}]_T). \tag{13}$$

3. In this section we examine the sequences Λ_i and $\{R_{in}\}$ given by (5). For the purposes of this section we may suppress the subscript i and consider the eigenvalue problem:

$$\begin{gathered} R'' + \frac{m-1}{r} R' + \left(\lambda - \frac{\nu^2}{r^2}\right) R = 0, \qquad 0 < r < 1, \\ R(0+) \text{ bounded}, \qquad R'(1) = 0, \end{gathered} \tag{5'}$$

in which ν appears as a parameter. We take $\nu \geqslant 1$, this being valid for the eigenvalues ν_i given by (4) (see, e.g., Ref. [2]). We seek the eigenvalues $\lambda = \lambda_1, \lambda_2, \ldots$ and eigenfunctions $R = R_1, R_2, \ldots$ satisfying (5′) with the normalization: $R(1) = 1$.

It is convenient to set $z_n = \sqrt{\lambda_n}$, in which case we can express z_n, R_n in terms of Bessel functions:

$$\begin{gathered} R_n(r) = r^{1-m/2} J_\nu(z_n r)/J_\nu(z_n), \\ \sqrt{\lambda_n} = z_n \text{ is the } n\text{–th zero of } J_\nu'. \end{gathered} \tag{14}$$

This leads to the formula

$$N_n^2 = \int_0^1 | R_n(r)|^2 r^{m-1}\, dr = \int_0^{z_n} J_\nu(z)^2\, z\, dz/\lambda_n J_\nu(z_n)^2.$$

On multiplying Bessel's equation by zJ_ν' and integrating by parts between 0 and z_n we now obtain the uniform estimate

$$N_n^2 = \tfrac{1}{2}(1 - \nu^2/\lambda_n) \leqslant \tfrac{1}{2}. \tag{15}$$

Note that this also shows that $\lambda_n \geqslant \nu^2 \geqslant 1$.

We proceed next to obtain a uniform lower bound for the separations $(z_{n+1} - z_n)$, namely, we show that $z_{n+1} - z_n \geqslant 1/2$. Note first that with the change of scale $r \mapsto y = rz_n$, $\hat{R}(y) = R_n(r)$, the differential equation of (5′) becomes

$$\hat{R}'' + \frac{m-1}{y}\hat{R}' + (1 - \nu^2/y^2)\hat{R} = 0. \tag{16}$$

Passing now to the polar phase plane variables (cf., Ref. [3])

$$y^{m-1}\hat{R}' = \rho\cos\theta, \qquad \hat{R} = \rho\sin\theta,$$

one obtains the system

$$\begin{aligned} \rho' &= [y^{1-m} - (y^{m-1} - \nu^2 y^{m-3})]\,\rho\sin\theta\cos\theta, \\ \theta' &= y^{1-m}\cos^2\theta + [y^{m-1} - \nu^2 y^{m-3}]\sin^2\theta. \end{aligned} \tag{17}$$

Note that the condition $\hat{R}'(z_n) = 0$ is equivalent to $\cos\theta(z_n) = 0$. For each interval $[z_n, z_{n+1}]$ we have $y \geqslant z_n \geqslant \nu \geqslant 1$ so $\theta' > 0$ and we may find a (unique) subinterval $(\alpha, \beta) \subset (z_n, z_{n+1}]$ for which

$$\begin{aligned} &\sin^2\theta(y) < y^{2-2m} && \text{for } \ y \in (\alpha, \beta), \\ &\sin^2\theta(\alpha) = \alpha^{2-2m}, && \sin^2\theta(\beta) = 0. \end{aligned}$$

On this subinterval, by (17), we have $\theta' \leqslant 2y^{1-m} \leqslant 2\alpha^{1-m}$ so that

$$\theta(\beta) - \theta(\alpha) = \int_\alpha^\beta \theta'\,dy \leqslant 2\alpha^{1-m}(\beta - \alpha).$$

But

$$\alpha^{1-m} = |\sin\theta(\beta) - \sin\theta(\alpha)| \leqslant \theta(\beta) - \theta(\alpha),$$

and this gives the desired lower bound for the separation:

$$\begin{aligned} z_{n+1} - z_n > \beta - \alpha &\geqslant [\theta(\beta) - \theta(\alpha)]/\max\{\theta'(y) : \alpha \leqslant y \leqslant \beta\} \\ &\geqslant [\sin\theta(\beta) - \sin\theta(\alpha)]/2\alpha^{1-m} = \tfrac{1}{2}\,. \end{aligned} \tag{18}$$

(Although (18) gives ***uniform*** estimates of the separation of zeroes of J_ν' for arbitrary $\nu \geqslant 1$, much more accurate ***asymptotic*** estimates are available (e.g., in Ref. [7]).)

4. The present section is devoted to obtaining the needed estimates for the quantities $\| l_{in}^T \|$ occurring in Section 2. In pursuing this goal we develop, as mentioned earlier, results for approximation by Dirichlet series which are uniform over certain classes of sequences.

We begin by referring to results of L. Schwartz [9, Chaps. 6 and 9] (see also Refs. [1], [4], [8], [10]). We restate these in a form convenient for our purpose. In what follows a *regular sequence* $\Lambda = (\lambda_1, \lambda_2, ...)$ is one for which $\sum_k 1/\lambda_k$ converges and for which there is a positive lower bound for the separations $(\lambda_{k+1} - \lambda_k)$.

THEOREM S_1. *If Λ is a regular sequence then each function $F \in [\Lambda] \subset L^2(0, \infty)$ posesses a unique Dirichlet series expansion of the form*

$$F(\tau) = \sum_n d_n e^{-\lambda_n \tau} \tag{19}$$

which converges for complex $\tau = t + i\sigma$ in the halfplane $t > 0$. Moreover, there exists a positive function C^Λ, defined on $(0, \infty)$ by

$$C^\Lambda(t) = \sqrt{2\pi} \sum_n \left(1 + \frac{\lambda_n}{2\pi}\right)^2 \prod_{m \neq n} \left| \frac{\lambda_n + \lambda_m}{\lambda_n - \lambda_m} \right| e^{-(\lambda_n - \lambda_1)t}, \tag{20}$$

which is bounded on each interval $[t_0, \infty)$, $t_0 > 0$, and for which

$$\sum_n \| l_n^\Lambda \| \, e^{-(\lambda_n - \epsilon)t} \leqslant C^\Lambda(t) \, e^{-(\lambda_1 - \epsilon)t} \qquad \text{for } \epsilon > 0. \tag{21}$$

Consequently, for each $F \in [\Lambda] \subset L^2(0, \infty)$,

$$e^{\epsilon t} \, |F(t + i\sigma)| < |F|_{(0,\infty)} \, C^\Lambda(t) \, e^{-(\lambda_1 - \epsilon)t}, \qquad \text{for all } 0 \leqslant \epsilon \leqslant \lambda_1. \tag{22}$$

THEOREM S_2. *If Λ is a regular sequence and $T > 0$, then every function $F \in [\Lambda]_T$ is the restriction to $(0, T)$ of a unique function $F^* \in [\Lambda]$, and there exists a positive constant c_T^Λ, independent of F, such that*

$$|F^*|_{(0,\infty)} \geqslant |F|_{(0,T)} \geqslant c_T^\Lambda \, |F^*|_{(0,\infty)} . \tag{23}$$

We now wish to consider classes of sequences which obey a stronger separation requirement than regularity. To wit, given $\beta > 1$ and $\delta > 0$, we consider all sequences Λ satisfying

$$\begin{aligned} \lambda_{n+1}^{1/\beta} - \lambda_n^{1/\beta} &\geqslant 2\delta, \qquad n \geqslant 1, \\ \lambda_1^{1/\beta} &\geqslant 2\delta. \end{aligned} \tag{24}$$

The regularity of such sequences is clear. Moreover by the results in Section 3, the sequences Λ_i determined by (5) satisfy (24) with $\delta = 1/4$, $\beta = 2$.

The following result is a uniform version of Theorems S_1 and S_2.

THEOREM 1. *Let $\mathscr{L}_{\delta,\beta}$ denote the class of all sequences Λ satisfying* (24) *for a prescribed pair δ, β with $\delta > 0$, $\beta > 1$. Then there exists a positive function $C_{\delta,\beta}$ on $(0, \infty)$, bounded on each interval $[t_0, \infty)$, $t_0 > 0$, such that*

$$e^{\epsilon t} \, |F(t + i\sigma)| \leqslant |F|_{(0,\infty)} \, C^{\Lambda}(t) \, e^{-(\lambda_1 - \epsilon) t} \leqslant |F|_{(0,\infty)} \, C_{\delta,\beta}(t),$$

when (25)

$$\Lambda \in \mathscr{L}_{\delta,\beta} \qquad \textit{and} \qquad 0 \leqslant \epsilon \leqslant \delta^{\beta}.$$

Moreover, for each $T > 0$ there exists a positive constant $c_{\delta,\beta,T}$ such that

$$c_T{}^{\Lambda} \geqslant c_{\delta,\beta,T} \qquad \textit{for all} \quad \Lambda \in \mathscr{L}_{\delta,\beta}. \tag{26}$$

Proof. Define a positive sequence $M = (\mu_1, \mu_2, \ldots)$ as follows

$$(\mu_j)^{1/\beta} = (j + 1)\,\delta, \qquad j \geqslant 1. \tag{27}$$

We will show that we may take

$$c_{\delta,\beta}(t) = C^M(t/2^{\beta}).$$

Since M is regular, C^M is, by Theorem S_1, bounded on each interval $[t_0, \infty)$, $t_0 > 0$. Hence it suffices to verify the inequality

$$\sup\{C^{\Lambda}(t) e^{-(\lambda_1 - \delta^{\beta}) t} : \Lambda \in \mathscr{L}_{\delta,\beta}\} \leqslant C^M(t/2^{\beta}) \, e^{-(\mu_1 - \delta^{\beta}) t/2^{\beta}}, \tag{28}$$

which by (20) is equivalent to

$$\begin{aligned} &\sum_n \left(1 + \frac{\lambda_n}{2\pi}\right)^2 \prod_{m \neq n} \left| \frac{\lambda_n + \lambda_m}{\lambda_n - \lambda_m} \right| e^{-(\lambda_n - \delta^{\beta}) t} \\ &\qquad \leqslant \sum_j \left(1 + \frac{\mu_j}{2\pi}\right)^2 \prod_{k \neq j} \left| \frac{\mu_j + \mu_k}{\mu_j - \mu_k} \right| e^{-(\mu_j - \delta^{\beta}) t/2^{\beta}} \end{aligned} \tag{29}$$

for all $\Lambda \in \mathscr{L}_{\delta,\beta}$.
Suppose $\Lambda \in \mathscr{L}_{\delta,\beta}$ is fixed. By (24) and (27) there exists for each n an index j_n such that

$$\lambda_n \in [\mu_{j_n - 1}, \mu_{j_n})$$

and $n \neq m$ implies $j_n \neq j_m$. In order to show that (29) holds it is sufficient to show that each term on the left is dominated by an appropriate term on the right. We show in fact that for each n and for $j = j_n$ one has

$$\begin{aligned} &\left(1 + \frac{\lambda_n}{2\pi}\right)^2 \prod_{m \neq n} \left| \frac{\lambda_n + \lambda_m}{\lambda_n - \lambda_m} \right| e^{-(\lambda_n - \delta^{\beta}) t} \\ &\qquad \leqslant \left(1 + \frac{\mu_j}{2\pi}\right)^2 \prod_{k \neq j} \left| \frac{\mu_j + \mu_k}{\mu_j - \mu_k} \right| e^{-(\mu_j - \delta^{\beta}) t/2^{\beta}}. \end{aligned} \tag{30}$$

Note that

$$\lambda_n - \delta^\beta \geqslant [(j-1)^\beta - 1]\,\delta^\beta \geqslant [(j^\beta - 2^\beta)/2^\beta]\,\delta^\beta = (\mu_j - \mu_1)/2^\beta.$$

Thus

$$\left(1 + \frac{\lambda_n}{2\pi}\right)^2 e^{-(\lambda_n - \delta^\beta)\,t} \leqslant \left(1 + \frac{\mu_j}{2\pi}\right)^2 e^{-(\mu_j - \delta^\beta)\,t/2^\beta},$$

and it remains only to compare the infinite products in (30). Now by (24) we have, for $k = 1, 2, \ldots$,

$$|\,\lambda_{n\pm k}^{1/\beta} - \lambda_n^{1/\beta}\,| \geqslant 2\delta k,$$

so, since $j = j_n \geqslant 2$ for all n,

$$\left(\frac{\lambda_{n-k}}{\lambda_n}\right)^{1/\beta} \leqslant \frac{(j-2k)\,\delta}{(j-1)\,\delta} \leqslant \frac{j-k}{j},$$

$$\left(\frac{\lambda_n}{\lambda_{n+k}}\right)^{1/\beta} \leqslant \frac{j\delta}{((j-1)+2k)\,\delta} \leqslant \frac{j}{j+k}, \qquad n \geqslant 1, \quad k \geqslant 1.$$

That is,

$$\frac{\lambda_{n-k}}{\lambda_n} \leqslant \frac{\mu_{j-k}}{\mu_j}, \qquad \frac{\lambda_n}{\lambda_{n+k}} \leqslant \frac{\mu_j}{\mu_{j+k}},$$

which implies that for both $m > n$ and $m < n$,

$$\left|\frac{\lambda_n + \lambda_m}{\lambda_n - \lambda_m}\right| \leqslant \left|\frac{\mu_j + \mu_{j+(m-n)}}{\mu_j - \mu_{j+(m-n)}}\right|.$$

On noting that any discarded factors of the right product in (30) are greater than 1, we see that (29) has been proved.

To complete the proof, we note that by Theorem S_2 there exists $c_T{}^\Lambda > 0$ for each $\Lambda \in \mathscr{L}$ and we set

$$c_T = c_{T,\delta,\beta} = \inf\{c_T{}^\Lambda : \Lambda \in \mathscr{L}\};$$

we need only prove that $c_T > 0$. If, indeed, we had $c_T = 0$, then there would exist a sequence Λ_k in $\mathscr{L}$ and a sequence of exponential polynomials $P_k \in sp\{e^{-\lambda t} : \lambda \in \Lambda_k\}$ such that

$$|\,P_k\,|_{(0,\infty)} = 1, \qquad k = 1, 2, \ldots, \tag{31a}$$

$$|\,P_k\,|_{(0,T)} \to 0, \qquad k \to \infty. \tag{31b}$$

By (22), and (25) we have

$$e^{\epsilon t}\,|\,P_k(t + i\sigma)| \leqslant C_{\delta,\beta}(t) \qquad \text{for} \quad t > 0, \qquad 0 \leqslant \epsilon \leqslant \delta^\beta, \tag{32}$$

where $C_{\delta,\beta}(t)$ is uniformly bounded on any interval $[t_0, \infty)$, $t_0 > 0$. Thus, a standard result on normal families of analytic functions implies the existence of a subsequence of $\{P_k\}$—which we again denote by $\{P_k\}$—which converges to an analytic limit function P. Since the convergence asserted is uniform in any compact subset of the right half-plane, it follows that $P_k \to P$ in, e.g., $L^2(T/2, T)$. By (31b), this shows $P = 0$ a.e. in $(T/2, T)$ whence, as P is analytic in the right half-plane, we conclude that $P = 0$. Now using (32) we also have $P_k \to P = 0$ in $L^2[T, \infty)$, so

$$| P_k |^2_{(0,\infty)} = | P_k |^2_{(0,T)} + | P_k |^2_{(T,\infty)} \to 0$$

(using (31b)), which contradicts (31a). Hence $c_T > 0$.

Remarks. (1) In the above theorem the class $\mathscr{L}_{\delta,\beta}$ was obtained by applying a map $\lambda_k = \varphi(z_k)$ (here, $\varphi(z) = z^\beta$) to uniformly separated sequences $(z_1, z_2, \ldots)$. In fact, $\mathscr{L}$ was specified by giving φ, (i.e., β) as well as the separation of the z's, and a lower bound for $z_1 > 0$. The proof of the theorem was tailored to the exponential nature of the mapping φ considered, but one might also consider more general mappings. It would seem of interest to determine reasonable sufficient conditions on φ for the analogous theorem to be valid.

(2) Although we have emphasized the spaces $L^2(0, \infty)$ and $L^2(0, T)$, analogous results hold for L^p spaces with $p \neq 2$. In fact Theorems S_1 and S_2 apply for $p \neq 2$ (cf. Ref. [9]) and a correspondingly revised proof for Theorem 1 is also available.

(3) It should be noted that if $\Lambda \in \mathscr{L}_{\delta,\beta}$ satisfies a restriction of the form

$$\lambda_1^{1/\beta} \geqslant (r_0 + 1)\,\delta,$$

then the comparison sequence could be taken as the following subsequence of M,

$$M_0 = (M_{r_0}, M_{r_0+1}, \ldots).$$

In that case, the limit on ϵ in (25) could be increased to $0 \leqslant \epsilon \leqslant (r_0 + 1)^\beta \delta^\beta$, while if ϵ is held fixed $C_{\delta,\beta}(t) \to 0$ as $r_0 \to \infty$.

5. At this point we are ready to state and prove the anticipated result concerning the 'observation and prediction' problem.

THEOREM 2. *The 'observation and prediction' problem is well-posed. That is, for any $T > 0$ the mapping*

$$A : u\,|_{(0,T)\times\partial\mathscr{R}} \in L^2((0, T) \times \partial\mathscr{R}) \mapsto u(\cdot, T) \in L^2(\mathscr{R})$$

is a well-defined, bounded (indeed, compact) linear map for solutions u of the Neumann problem (1) *for the heat equation on the m-ball.*

Proof. For $m = 1$ this follows from the principal result of Ref. [6]. For $m > 1$ we may continue to assume $\mathscr{R}$ is the unit m-ball and use the expansion (3). As noted in Section 3, the sequences Λ_i, $\{R_{in}\}$ obtained from (5′) have $\nu = \nu_i \geqslant 1$. Thus, we have the uniform bound $N_{in}^2 \leqslant 1/2$, while by (15) and (18) each sequence Λ_i is in $\mathscr{L}_{\delta,\beta}$ with $\delta = 1/4$, $\beta = 2$. Applying Theorem 1 together with (21) gives

$$e^{\epsilon t} \sum_n \| l_{in} \| e^{-\lambda_{in} t} \leqslant C_{\delta,\beta}(t), \qquad 0 \leqslant \epsilon \leqslant (\tfrac{1}{4})^2. \tag{33}$$

Noting Theorem S_2 we have

$$l_{in}^T(F) = l_{in}(F^*),$$

so, using (23) and Theorem 1,

$$\| l_{in}^T \| \leqslant \| l_{in} \| / c_{\delta,\beta,T} \,. \tag{34}$$

Substituting (34) into (33), squaring and discarding cross terms gives

$$\sum_n \| l_{in}^T \|^2 e^{-2\lambda_{in} T} \leqslant e^{-2\epsilon T} C_{\delta,\beta}^2(T)/c_{\delta,\beta,T}^2$$

for each i. This, in view of (11) and the bound on N_{in}^2, gives a bound for A :

$$\| A \| \leqslant e^{-\epsilon T} C_{\delta,\beta}(T)/\sqrt{2}\, c_{\delta,\beta,T} \,. \tag{35}$$

To show that A is compact we observe that the map $f \mapsto w$ can be factored as

$$f \mapsto f\,|_{t<T'} \mapsto u(T', \cdot) \mapsto u(T, \cdot) = w$$

with $0 < T' < T$. The restriction map is certainly bounded; the second map is of the sort shown above to be bounded, with T' for T; the third is the solution operator corresponding to the initial value problem for (1) and this is well-known to be compact. Thus, as a product of bounded maps with a compact map, A is compact. An alternative proof could be given, using Remark 3 following Theorem 1 and the fact that $\nu_i \to \infty$, to show directly that A is the limit in norm of operators of finite rank.

COROLLARY. *The 'observation and prediction' problem is well-posed (indeed, the linear mapping* $A : f \to w$ *is compact) when the codomain is taken to be* $L^p(1 \leqslant p \leqslant \infty)$ *and the domain* $\mathscr{M}$ *is topologized as a subset of* $L^r(2 \leqslant r \leqslant \infty)$.

Proof. This follows from the theorem and two observations:

(1) In considering the factorization of A given in the proof above, it is easily seen that the solution mapping $u(T', \cdot) \to u(T, \cdot)$, for the initial value problem is actually compact from $L^2(\mathscr{R})$ to $L^p(\mathscr{R})$ for all $p \geqslant 1$.

(2) Since $(0, T) \times \partial\mathscr{R}$ has finite measure, the inclusion map from $L^r((0, T) \times \partial\mathscr{R})$ to $L^2(0, T) \times \partial\mathscr{R}$ is continuous for $r \geqslant 2$.

6. We now wish to summarize the work of the preceding sections in more general terms.

Let $\{H_p\}$ be a complete set of (orthonormalized) eigenfunctions, indexed in some order, for the Laplacian for the region $\mathscr{R}$ with the Neumann boundary condition (1b); let $\{\lambda_p\}$ be the corresponding eigenvalues and set $e_p(t) = \exp[-\lambda_p t]$. Then, analogously to (3), any solution of (1) has an expansion of the form

$$u(t, x) = \sum_p c_p e_p(t) H_p(x), \qquad t > 0, \quad x \in \mathscr{R}, \tag{36}$$

so that

$$\begin{aligned} A : f = \sum_p c_p e_p h_p \in L^2((0, T) \times \partial\mathscr{R}) \mapsto \\ w = \sum_p c_p e_p(T) H_p \in L^2(\mathscr{R}), \end{aligned} \tag{37}$$

where

$$h_p = H_p \mid_{\partial\mathscr{R}} .$$

Clearly, if A is continuous, then the linear functionals m_p on $\mathscr{M}$ given by

$$m_p : f \to c_p$$

must be well-defined and continuous since, by the orthonormality of $\{H_p\}$,

$$m_p(f) = c_p = \langle Af, H_p \rangle_{\mathscr{R}} / e_p(T).$$

Conversely, we may express A in terms of these functionals:

$$Af = \sum_p m_p(f)\, e_p(T)\, H_p ,$$

so that continuity of A follows from having suitable estimates for the quantities $\| m_p \|$.

Our approach, in this paper and in Ref. [5], has been to partition the set $\{p\}$ of indices into subsets $\pi_i (i = 1, 2, \ldots)$. Letting

$$\mathcal{M}_i = \overline{sp}\left\{\sum_{p\in\pi_i} c_p e_p h_p\right\} \subset \mathcal{M} \subset L^2((0, T) \times \partial\mathcal{R}),$$

we may consider the restrictions $\hat{m}_p$ of m_p to $\mathcal{M}_i$ $(p \in \pi_i)$ and write

$$f = \sum_i f_i = \sum_i \left(\sum_{p\in\pi_i} c_p e_p h_p\right),$$

$$Af = \sum_i A_i f_i = \sum_i \left(\sum_{p\in\pi_i} \hat{m}_p(f_i)\, e_p(T)\, H_p\right), \qquad f \in \mathcal{M}_i,$$

where A_i is the restriction of A to $\mathcal{M}_i$. Then

$$\begin{aligned} | Af |_{\mathcal{R}}^2 &= \sum_i | A_i f_i |_{\mathcal{R}}^2 \\ &= \sum_i \sum_{p\in\pi_i} | \hat{m}_p(f_i)\, e_p(T)|^2. \end{aligned} \tag{38}$$

Hence

$$\begin{aligned} | Af |_{\mathcal{R}}^2 &\leqslant \sum_i \left(\sum_{p\in\pi_i} \| \hat{m}_p \|^2 e_p(2T)\right) | f_i |_{(0,T)\times\partial\mathcal{R}}^2 \\ &\leqslant \sup_i \left\{\sum_{p\in\pi_i} \| \hat{m}_p \|^2 e_p(2T)\right\} \sum_i | f_i |_{(0,T)\times\partial\mathcal{R}}^2 . \end{aligned} \tag{39}$$

In the case of the m-ball, the subspaces $\mathcal{M}_i$ were actually orthogonal (each being associated with a single Ω_i) so

$$\sum_i | f_i |_{(0,T)\times\partial\mathcal{R}}^2 = | f |_{(0,T)\times\partial\mathcal{R}}^2 \tag{40}$$

and it follows from this and (39) that

$$\| A \|^2 \leqslant \sup_i \left\{\sum_{p\in\pi_i} \| \hat{m}_p \|^2 e_p(2T)\right\}. \tag{41}$$

In order to estimate the quantities $\| \hat{m}_p \|$ appearing in (41) we observe that the $\{\hat{m}_p\}$ can be expressed in terms of functionals $\{l_p^{A,T}\}$ of the type discussed in Section 4. In fact, for any q

$$\langle f_i , h_q \rangle_{\partial\mathcal{R}} (t) = \sum_{p\in\pi_i} c_p \langle h_p , h_q \rangle_{\partial\mathcal{R}}\, e_p(t),$$

and hence

$$\hat{m}_q(f_i) = c_q = l_q^T(\langle f_i, h_q\rangle_{\partial\mathscr{R}}(t))/|h_q|^2_{\partial\mathscr{R}}, \qquad q \in \pi_i, \tag{42}$$

where l_q^T is the functional which selects the coefficient of e_q for elements of

$$[\Lambda_i]_T = \overline{sp}\{e_p : p \in \pi_i\} \subset L^2(0, T).$$

Equation (42) leads to the appraisal

$$\| \hat{m}_p \| \leqslant \| l_p^T \| / | h_p |_{\partial\mathscr{R}}$$

so that (41) implies (compare (11)):

$$\| A \|^2 \leqslant \sup_i \left\{ \sum_{p\in\pi_i} \| l_p^T \|^2 / | h_p |^2_{\partial\mathscr{R}}\, e_p(2T) \right\}. \tag{43}$$

Now by (15),

$$| h_p |^{-2}_{\partial\mathscr{R}} = N_{in}^2 \leqslant \tfrac{1}{2}. \tag{44}$$

Hence, (43) implies

$$\| A \| \leqslant \sup_i \left\{ \sum_{p\in\pi_i} \| l_p^T \| \, e_p(T) \right\} \Big/ \sqrt{2},$$

which on applying Theorem 1 leads to the appraisal (see (35)):

$$\| A \| \leqslant e^{-\epsilon T} C_{\delta,\beta}(T)/\sqrt{2}\, c_{\delta,\beta,T}, \qquad 0 \leqslant \epsilon \leqslant (\tfrac{1}{4})^2.$$

In the case of 'cylindrical' sets, considered in Ref. [5], the spaces $\mathscr{M}_i$ are by no means orthogonal, so (40) fails to hold. We succeeded in that case by noting that there is available the following replacement for (40):

$$\sum_i | f_i |^2_{(0,T)\times\mathscr{B}} = | f |^2_{(0,T)\times\mathscr{B}} \leqslant | f |^2_{(0,T)\times\partial\mathscr{R}}, \tag{45}$$

which stems from the orthogonality of the $f_i |_{(0,T)\times\mathscr{B}}$, where $\mathscr{B} = \{0\} \times \mathscr{D}$ is the base of the cylinder $\mathscr{R} = [0, 1] \times \mathscr{D}$. Correspondingly, (42) and (44) were replaced by

$$\hat{m}_q(f_i) = l_q^T(\langle f_i, h_q\rangle_{\mathscr{B}}(t))/|h_q|^2_{\mathscr{B}},$$

$$| h_q |^2_{\mathscr{B}} = 1.$$

Another, more powerful, approach which seems more likely to extend to the case of a general class of regions $\mathscr{R}$, involves the replacement of (40) by a condition of the form

$$\sum_i |f_i|^2_{(T',T)\times\partial\mathscr{R}} \leqslant K^2 |f|^2_{(0,T)\times\partial\mathscr{R}} \tag{46}$$

for some choice of $T' \in (0, T)$ and a corresponding constant K. In this situation we are led to appraise the quantities $\{\hat{m}_p\}$ not in terms of the functionals $\{l_p{}^T\}$ used earlier, but rather in terms of the functionals $\{l_p^{T_1}\}$ where $T_1 = T - T'$. Note first that the function $g_i = \langle f_i , h_q\rangle_{\partial\mathscr{R}}$ given by

$$\langle f_i , h_q\rangle_{\partial\mathscr{R}}(t) = \sum_{p\in\pi_i} c_p\langle h_p , h_q\rangle_{\partial\mathscr{R}}\, e_p(t),\ldots$$

when restricted to the interval $T' < t < T$, corresponds by a translation of variable to the following function on $(0, T_1)$:

$$g_i(T' + \tau) = \sum_{p\in\pi_i} c_p\langle h_p , h_q\rangle_{\partial\mathscr{R}}\, e_p(T')\, e_p(\tau), \qquad 0 < \tau < T_1 .$$

Hence we have the formula, parallel to (42),

$$\hat{m}_q(f_i) = c_q = l_q^{T_1}(g_i(T' + \tau))/[| h_q |^2 e_q(T')], \tag{47}$$

where $l_q{}^{T_1}$ selects the coefficient of e_q in elements of

$$[\Lambda_i]_{T_1} = \overline{sp}\{e_p : p \in \pi_i\} \subset L^2(0, T_1).$$

This leads to the appraisal

$$\begin{aligned} | \hat{m}_q(f_i)| &\leqslant \| l_q^{T_1} \| \, | g_i(T' + \tau)|_{(0,T_1)}\, e_q(- T')/| h_q |^2_{\partial\mathscr{R}} \\ &\leqslant \| l_q^{T_1} \| \, |\langle f_i , h_q\rangle|_{(T',T)}\, e_q(- T')/| h_q |^2_{\partial\mathscr{R}} \\ &\leqslant \| l_q^{T_1} \| \, | f_i |_{(T',T)\times\partial\mathscr{R}}\, e_q(- T')/| h_q |_{\partial\mathscr{R}} . \end{aligned}$$

Hence (38) implies

$$\begin{aligned} | Af |^2_{\mathscr{R}} &\leqslant \sum_i \sum_{p\in\pi_i} \| l_p^{T_1} \|^2 | f_i |^2_{(T',T)\times\partial\mathscr{R}}\, e_p(2(T - T'))/| h_p |^2_{\partial\mathscr{R}} \\ &\leqslant \sum_i \Big(\sum_{p\in\pi_i} \| l_p^{T_1} \|^2/| h_p |^2_{\partial\mathscr{R}}\, e_p(2T_1)\Big) | f_i |^2_{(T',T)\times\partial\mathscr{R}} \qquad (48) \\ &\leqslant \sup_i \Big\{ \sum_{p\in\pi_i} \| l_p^{T_1} \|^2/| h_p |^2_{\partial\mathscr{R}}\, e_p(2T_1)\Big\} \sum_i | f_i |^2_{(T',T)\times\partial\mathscr{R}} . \end{aligned}$$

Combining this with (46) we deduce that

$$\| A \|^2 \leqslant K^2 \sup_i \left\{ \sum_{p \in \pi_i} \| l_p^{T_1} \|^2 / | h_p |_{\partial \mathscr{R}}^2 \, e_p(2T_1) \right\}$$

$$\leqslant K^2 \sup_i \left\{ \left[\sum_{p \in \pi_i} \| l_p^{T_1} \| \, e_p(T_1/2) \right] \sup_{p \in \pi_i} \frac{e_p(T_1/2)}{| h_p |_{\partial \mathscr{R}}} \right\}^2 .$$

Therefore, under the very mild condition that, say,

$$| h_p |_{\partial \mathscr{R}}^{-1} \, e_p(T_1/2) < K_1 < \infty, \qquad \text{for all } p, \tag{49}$$

we have

$$\| A \| \leqslant K K_1 \sup_i \left\{ \sum_{p \in \pi_i} \| l_p^{T_1} \| \, e_p(T_1/2) \right\} .$$

Finally, assuming that the partitions are such that the sequences

$$\Lambda_i = \{ \lambda_p : p \in \pi_i \}$$

all belong to some $\mathscr{L}_{\delta,\beta}$, so that Theorem 1 is applicable, we have the desired boundedness assertion:

$$\| A \| \leqslant K K_1 C_{\delta,\beta}(T_1/2) / c_{\delta,\beta,T_1} . \tag{50}$$

We have shown above that the inequality in (46) is *sufficient* for boundedness of A. What is particularly striking is that it is also *necessary* if A is to be bounded for all choices of $T > 0$. More precisely, we have the following result:

THEOREM 3. *Suppose that the functions $h_p = H_p \,|_{\partial \mathscr{R}}$ have the property that for each $\epsilon > 0$*

$$\{ | h_p |_{\partial \mathscr{R}}^{-1} \, e_p(\epsilon) \} \textit{ is bounded.} \tag{51}$$

Suppose also that for a given $T > 0$ there exists a partition of $\{\lambda_p\}$ into subsets

$$\Lambda_i = \{ \lambda_p : p \in \pi_i \}$$

satisfying:

(H1) *$\Lambda_i \in \mathscr{L}_{\delta,\beta}$ for some $\delta > 0$, $\beta > 1$ and all i;*

(H2) *For some $T' < T$ there are associated to every*

$$f \in \overline{sp}\{h_p e_p\} \subset L^2((0, T) \times \partial \mathscr{R}) \qquad \textit{functions}$$

$$f_i \in \overline{sp}\{h_p e_p : p \in \pi_i\} \subset L^2((T', T) \times \partial \mathscr{R}) \qquad \textit{satisfying}$$

$$\sum f_i = f \,|_{(T',T) \times \partial \mathscr{R}} \in L^2((T', T) \times \partial \mathscr{R}).$$

Then the following relation holds:

$$\sum |f_i|^2_{(T',T)\times\partial\mathscr{R}} \leqslant K\,|f|^2_{(0,T)\times\partial\mathscr{R}}, \tag{i}$$

and the solution operator

$$A : f \to u(T, \cdot)$$

is bounded.

Conversely, if the functions $\{h_p\}$ *have the property that for each* $\epsilon > 0$,

$$\sum_p |h_p|_{\partial\mathscr{R}}\, e_p(\epsilon) \text{ converges,} \tag{52}$$

then the boundedness for every $T > 0$ *of the solution operator*

$$A : f \to u(T, \cdot)$$

implies that (i) *is valid for every such partition of* $\{\lambda_p\}$ *and every* $T > T' > 0$.

Proof. The proof of the first assertion is as given above once we notice that (i) follows from (H2) by the uniform boundedness principle.

In order to verify the converse we note that the expressions

$$f_i = \sum_{p\in\pi_i} c_p h_p e_p\,, \qquad \text{with} \qquad c_p = \langle Af, H_p\rangle_{\mathscr{R}}\, e_p(-T),$$

satisfy

$$\begin{aligned}
|f_i|_{(T',T)\times\partial\mathscr{R}} &\leqslant \Big[\sum_{p\in\pi_i} |c_p|\,|h_p|_{\partial\mathscr{R}}\,|e_p|_{(T',T)}\Big]^2 \\
&\leqslant \Big[\sum_{p\in\pi_i} |c_p|\,|h_p|_{\partial\mathscr{R}}\, e_p(T')\,|e_p|_{(0,T_1)}\Big]^2 \\
&\leqslant \sum_{p\in\pi_i} [|c_p|\, e_p(T'/2)]^2 \sum_{p\in\pi_i} [|h_p|_{\partial\mathscr{R}}\,|e_p|_{(0,T_1)}\, e_p(T'/2)]^2.
\end{aligned}$$

Since $|e_p|_{(0,T_1)} \leqslant |e_p|_{(0,\infty)} = 1/\sqrt{2\lambda_p}$, it is easily seen by (52) that the second factor on the right side is bounded, uniformly in i, by a constant D^2. On summing over i we obtain

$$\sum_i |f_i|^2_{(T',T)\times\partial\mathscr{R}} \leqslant D^2 \sum_p [|c_p|\, e_p(T'/2)]^2$$

which, by the formula (38) for $|A'f|^2$, where

$$A' = f|_{(0,T'/2)\times\partial\mathscr{R}} \mapsto u(T'/2, \cdot) \in L^2(\mathscr{R}), \textit{ yields}$$

$$\sum_i |f_i|^2_{(T',T)\times\partial\mathscr{R}} \leqslant D^2\, \|A_{T'/2}\|^2\, |f|^2_{(0,T'/2)\times\partial\mathscr{R}}\,,$$

as was claimed.

In view of Theorem 3 we have been led to make the following conjecture:

Conjecture. For each piecewise C^∞ region $\mathscr{R}$, the sequence $\{h_p\}$ satisfies (51) and the set $\{\lambda_p\}$ can be partitioned into subsequences $\Lambda_i = \{\lambda_p : p \in \pi_i\}$ satisfying both (H1) and (H2) of Theorem 3.

It is clear that the "observation and prediction" problem will be well posed provided this conjecture holds, but we have been unable as yet to ascertain the truth of the conjecture.

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